8a. Uncertainty

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Overview

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Different websites show different probabilities, but do we know the true, objective probability that it rains today?

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Modern concepts of prob. enabled by *subjective probability*.

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Modern concepts of prob. enabled by *subjective probability*.

Today: modelling choice under *uncertainty* and getting subjective beliefs *from* choice.

Extraordinarily important for theory and — especially — for applications.

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1. Uncertainty
2. Subjective Expected Utility
3. Savage’s Framework
4. Uncertainty Aversion
5. More

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2. Subjective Expected Utility
   - Anscombe-Aummann Framework
   - State-Dependent SEU
   - State-Independent SEU

3. Savage's Framework

4. Uncertainty Aversion

5. More

Anscombe-Aummann Framework

Main ingredients

- Ω: set of states of the world, finite;
- X: set of consequences or outcomes, finite;
- $f : \Omega \to \Delta(X)$: an act;
- $\mathcal{F} := \Delta(X)^{\Omega}$: set of acts;
- $\succsim \subseteq \mathcal{F}^2$: preference relation.

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Two Sources of Uncertainty

(i) subjective uncertainty (horse race): which state $\omega \in \Omega$ will be realised

(ii) objective uncertainty/risk (roulette wheel): which consequence $x \in X$ will be realised in a lottery.

Can be seen as compound lottery: a potentially different objective lottery is triggered by each (uncertain) state of the world

Anscombe-Aummann Framework

Goal: characterise exact properties of $\succsim$ that enable SEU representation

Definition

$\succsim$ admits a SEU representation if $\exists u : X \to \mathbb{R}$ and $\mu \in \Delta(\Omega)$ s.t. $\forall f, g \in \mathcal{F}$, $f \succsim g \iff \mathbb{E}_{\mu}[\mathbb{E}_{f(\omega)}[u]] \geq \mathbb{E}_{\mu}[\mathbb{E}_{g(\omega)}[u]]$.

- Recover (i) Bernoulli utility function on consequences, $u : X \to \mathbb{R}$, *and* (ii) probability measure $\mu \in \Delta(\Omega)$.
- For given state ω, $f(\omega)$ is objective prob. distrib. in $\Delta(X) \implies \mathbb{E}_{f(\omega)}[u]$ is vNM EU
- μ represents (as if) DM's belief over states; take expectations of vNM EU wrt μ to get *subjective* expected utility
- Importantly: want to pin down DM's beliefs from preferences

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More Definitions

Mixture $\alpha f + (1 - \alpha)g \in \mathcal{F}$, $f, g \in \mathcal{F}$ and $\alpha \in [0, 1]$

$$(\alpha f + (1 - \alpha)g)(\omega) = \alpha f(\omega) + (1 - \alpha)g(\omega)$$

$\forall p \in \Delta(X)$, denote **constant act** $\tilde{p} \in \mathcal{F}$ s.t. $\tilde{p}(\omega) = p \in \Delta(X)$ for every $\omega \in \Omega$.

State-Dependent SEU

Definition

- $\succsim$ sat. **continuity** iff $\{f_n, g_n\}_n \subset \mathcal{F} : f_n \succsim g_n \forall n$ and $(f_n, g_n) \to (f, g), \implies f \succsim g$.
- $\succsim$ sat. **independence** iff $\forall f, g, h \in \mathcal{F}$ and $\forall \alpha \in (0, 1]$, $f \succsim g \iff \alpha f + (1 - \alpha)h \succsim \alpha g + (1 - \alpha)h$.

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Definition

$\succsim$ admits a state-dependent SEU representation if $\exists u : X \times \Omega$ and $\mu \in \Delta(\Omega)$ s.t. $\forall f, g \in \mathcal{F}$, $f \succsim g \iff \mathbb{E}_\mu[\mathbb{E}_{f(\omega)}[u]] \geq \mathbb{E}_\mu[\mathbb{E}_{g(\omega)}[u]]$.

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An intermediate result:

Theorem

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A problem: we have $u(x, \omega)$, not $u(x)$.

Unable to separate preference over consequences and beliefs about states.

$u(x, \omega)$ state-*dependent* utility, capturing both preferences over consequences and beliefs about states $\implies$ uniform belief/prior μ, void of any empirical content.

Anscombe-Aummann Framework

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More Notation/Terminology

- $E \subseteq \Omega$: an event
- fEg: a 'conditional act,' where for acts f, g and event E, $fEg \in \mathcal{F}$ is such that $(fEg)(\omega) = f(\omega)$ is $\omega \in E$ and $(fEg)(\omega) = g(\omega)$ if otherwise;

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- Null event E: event s.t. $\forall f, g, h \in \mathcal{F} : f \succ g, fEh \sim gEh$ (why is it called null?);

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Lemma

Let $V : \mathcal{F} \to \mathbb{R}$ be affine and continuous. Then, $\forall \omega \in \Omega$, $\exists$ affine and continuous function $V_\omega : \Delta(X) \to \mathbb{R}$ s.t. $V(f) = \sum_\omega V_\omega(f(\omega)), \forall f \in \mathcal{F}$.

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Fix $f^* \in \mathcal{F}$.

- For any $f \in \mathcal{F}$, as V is affine:

$$\frac{1}{|\Omega|} f = \frac{1}{|\Omega|} f^* + \frac{1}{|\Omega|} \sum_\omega (f\{\omega\}f^* - f^*)$$

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&\iff V(f) = \sum_\omega \left[V\left((f\{\omega\}f^*)\right) - (|\Omega| - 1)\, V(f^*)/|\Omega|\right],
\end{aligned}$$

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- V is continuous and affine $\implies$ V_ω is continuous and affine. $\square$

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Theorem

A pref. rel. $\succsim$ on $\mathcal{F}$ sat. continuity and independence $\iff$ $\succsim$ admits a state-dependent SEU representation.

Proof

- $\forall f \in \mathcal{F}$, let $\rho_f \in \Delta(X \times \Omega)$ be s.t. $\rho_f(\{x\} \times \{\omega\}) := \frac{1}{|\Omega|} f(\omega)(x)\ \forall (x, \omega)$

  Note: ρ_f is joint distribution over $X \times \Omega$ with unif. marginal over Ω.

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- Define (i) $R := \{\rho_f \mid f \in \mathcal{F}\} \subseteq \Delta(X \times \Omega)$, set of all such joint distributions (convex, subset of $(|X \times \Omega| - 1)$ simplex).

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  (ii) $\trianglerighteq \subseteq R^2$ s.t. $\rho_f \trianglerighteq \rho_g \iff f \succsim g$.

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- $\succsim$ satisfies independence $\implies$ $\forall \alpha \in (0, 1]$ and $\forall f, g, h \in \mathcal{F}$, $\rho_f \trianglerighteq \rho_g \iff \rho_{\alpha f+(1-\alpha)h} \trianglerighteq \rho_{\alpha g+(1-\alpha)h}$, which implies:

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  $\implies$ $\trianglerighteq$ sat. independence.
- Adapted vNM EU representation theorem: $\exists\, v : X \times \Omega \to \mathbb{R}$ s.t.
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- By Lemma: $\exists V_\omega : \Delta(X) \to \mathbb{R}$ affine and continuous s.t. $V(f) = \sum_\omega V_\omega(f(\omega))$.
- Define $u : X \times \Omega$ as $u(x, \omega) := V_\omega(\delta_x)|\Omega|$.

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- $\trianglerighteq \subseteq R^2$ s.t. $\rho_f \trianglerighteq \rho_g \iff f \succsim g$.
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- Define $V : \mathcal{F} \to \mathbb{R}$ such that $V(f) := \mathbb{E}_{\rho_f}[v]$, affine and continuous.
- V represents $\succsim$: $f \succsim g \iff \rho_f \trianglerighteq \rho_g \iff V(f) \geq V(g)$.
- By Lemma: $\exists V_\omega : \Delta(X) \to \mathbb{R}$ affine and continuous s.t. $V(f) = \sum_\omega V_\omega(f(\omega))$.
- Define $u : X \times \Omega$ as $u(x, \omega) := V_\omega(\delta_x)|\Omega|$.
- For each ω, V_ω is affine, $V_\omega(p) = \sum_x p(x) \frac{1}{|\Omega|} u(x, \omega)$.

State-Dependent SEU

Theorem

A pref. rel. $\succsim$ on $\mathcal{F}$ sat. continuity and independence $\iff$ $\succsim$ admits a state-dependent SEU representation.

Proof

- $\forall f \in \mathcal{F}$, let $\rho_f \in \Delta(X \times \Omega)$ be s.t. $\rho_f(\{x\} \times \{\omega\}) := \frac{1}{|\Omega|} f(\omega)(x)\ \forall(x, \omega)$
- $\trianglerighteq \subseteq R^2$ s.t. $\rho_f \trianglerighteq \rho_g \iff f \succsim g$.
- $\exists\ v : X \times \Omega \to \mathbb{R}$ s.t. $\rho_f \trianglerighteq \rho_g \iff \mathbb{E}_{\rho_f}[v] \geq \mathbb{E}_{\rho_g}[v]$.
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- For each ω, V_ω is affine, $V_\omega(p) = \sum_x p(x) \frac{1}{|\Omega|} u(x, \omega)$.
- $V(f) = \sum_\omega V_\omega(f(\omega)) = \sum_\omega \sum_x f(\omega)(x) \frac{1}{|\Omega|} u(x, \omega) = \mathbb{E}_\mu[\mathbb{E}_{f(\omega)}[u]]$ with unif. $\mu \in \Delta(\Omega)$. $\square$

State-Independent SEU

Definition

$\succsim$ sat. **separability** iff $\forall p, q \in \Delta(X), h \in \mathcal{F}, \omega, \omega' \in \Omega$: $\{\omega\}$ and $\{\omega'\}$ are non-null events, $\tilde{p}\{\omega\}h \succsim \tilde{q}\{\omega\}h \iff \tilde{p}\{\omega'\}h \succsim \tilde{q}\{\omega'\}h$.

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Theorem

(1) A pref. rel. $\succsim$ on $\mathcal{F}$ sat. continuity, independence, and separability $\iff$ $\succsim$ admits a SEU representation.

(2) Moreover, u is unique up to positive affine transformations and, if $\exists f, g \in \mathcal{F} : f \succ g$, μ is unique.

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Theorem

(1) A pref. rel. $\succsim$ on $\mathcal{F}$ sat. continuity, independence, and separability $\iff$ $\succsim$ admits a SEU representation.

Proof

- Focus on $\implies$. Start with state-dependent SEU: $\exists u : X \times \boldsymbol{\Omega} \to \mathbb{R}$ s.t. $f \succsim g \iff \sum_{\omega,x} f(\omega)(x)u(x,\omega) \geq \sum_{\omega,x} g(\omega)(x)u(x,\omega)$.
- Let $U : \Delta(X) \times \boldsymbol{\Omega} \to \mathbb{R}$ be defined as $U(p,\omega) := \sum_{x \in X} p(x)u(x,\omega)$ for all $\omega \in \boldsymbol{\Omega}$, $p \in \Delta(X)$.

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Proof

- Take $p, q \in \Delta(X)$ and non-null $\{\omega\}$, $\omega \in \Omega$ s.t. $U(p, \omega) \geq U(q, \omega)$.
- Separability $\implies$ $\forall$ non-null $\{\omega'\}$, $\omega' \in \Omega$, $h \in \mathcal{F}$,

$$U(p, \omega) \geq U(q, \omega) \iff U(p, \omega) + \sum_{\omega'' \in \Omega \setminus \{\omega\}} U(h(\omega''), \omega'') \geq U(q, \omega) + \sum_{\omega'' \in \Omega \setminus \{\omega\}} U(h(\omega''), \omega'')$$

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- $\forall$ non-null ω, $\exists \alpha_\omega > 0, \beta_\omega : u(\cdot, \omega) = \alpha_\omega u + \beta_\omega$.
- Define $\mu \in \Delta(\Omega) : \mu(\omega) = 1_{\omega \in \Omega^*} \frac{\alpha_\omega}{\sum_{\omega' \in \Omega^*} \alpha_{\omega'}}$

$$f \succsim g \iff \mathbb{E}_\mu[\mathbb{E}_{f(\omega)}[u]] \geq \mathbb{E}_\mu[\mathbb{E}_{g(\omega)}[u]].$$

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Theorem

(2) Moreover, u is unique up to positive affine transformations and, if $\exists f, g \in \mathcal{F} : f \succ g$, μ is unique.

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Proof

- $u(x, \omega)$ is unique up to positive affine transformations $\implies$ $u(x)$ is too.
- Suppose $f \succ g$; WTS $!\mu$. Let $\nu \in \Delta(\Omega) : f \succsim g \iff \mathbb{E}_\nu[\mathbb{E}_{f(\omega)}[u]] \geq \mathbb{E}_\nu[\mathbb{E}_{g(\omega)}[u]]$.

$$f \succ g \implies u \text{ is non-constant} \quad (\text{ow } \mathbb{E}_\mu[\mathbb{E}_{f(\omega)}[u]] = \mathbb{E}_\mu[\mathbb{E}_{g(\omega)}[u]])$$
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- For $\omega' \in \Omega^*$, take (i) non-constant acts $\tilde{\delta}_z\{\omega'\}\tilde{\delta}_y$, and
  (ii) constant acts $h = \mu(\omega')\tilde{\delta}_z + (1 - \mu(\omega'))\tilde{\delta}_y$.

$$\mathbb{E}_\mu[\mathbb{E}_{(\tilde{\delta}_z\{\omega'\}\tilde{\delta}_y)(\omega)}[u]] = \mu(\omega')u(z) + (1 - \mu(\omega'))u(y) = \mathbb{E}_{\mu(\omega')\delta_z + (1-\mu(\omega'))\delta_y}[u] = \mathbb{E}_\mu[\mathbb{E}_{h(\omega)}[u]]$$

$$\iff \tilde{\delta}_z\{\omega'\}\tilde{\delta}_y \sim h \iff$$

$$\mathbb{E}_\nu[\mathbb{E}_{(\tilde{\delta}_z\{\omega'\}\tilde{\delta}_y)(\omega)}[u]] = \nu(\omega')u(z) + (1 - \nu(\omega'))u(y) = \mu(\omega')u(z) + (1 - \mu(\omega'))u(y) = E_\nu[\mathbb{E}_{h(\omega)}[u]]$$

$$\iff \mu(\omega') = \nu(\omega') \forall \omega' \in \Omega^*. \qquad \square$$

Subjective Expected Utility

Foundational result.

Unclear if state-dependence is feature of real-world or due to imprecise specification specification of consequences

$\mathcal{F}$	Ω	
	rain	no rain
taking umbrella	not wet but carrying umbrella	not wet, carrying umbrella
not taking an umbrella	wet, not carrying umbrella	not wet, not carrying umbrella

X ={having to carry an umbrella, not having to carry an umbrella}$\times${getting wet, not getting wet}

Monotonicity

Theorem

Pref. rel. $\succsim$ on $\mathcal{F}$ sat. **monotonicity** iff $\forall f, g \in \mathcal{F}, \tilde{f}(\omega) \succsim \tilde{g}(\omega)\ \forall \omega \in \mathbf{\Omega}$ s.t. $\{\omega\}$ is non-null $\implies f \succsim g$.

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Proposition

If $\succsim$ sat. independence and continuity, then separability and monotonicity are equivalent.

Overview

1. Uncertainty
2. Subjective Expected Utility
3. Savage’s Framework
4. Uncertainty Aversion
5. More

Savage's Framework

Savage 1954, *The Foundations of Statistics*.

Original formulation of SEU

- Ω: set of states of the world;
- X: set of consequences or outcomes;
- $f : \Omega \to X$: an act;
- $\mathcal{F} := X^{\Omega}$: set of acts;
- $\succsim \subseteq \mathcal{F}^2$: preference relation.

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Crucial difference: acts map to consequences; no need for state-dependent lotteries on the set of consequences.

Savage: $\mathbb{E}_{\mu}[u(f(\omega))] = \int_{\Omega} u(f(\omega))d\mu(\omega)$;

Anscombe–Aumann: $\mathbb{E}_{\mu}[\mathbb{E}_{f(\omega)}[u]] = \int_{\Omega} \int_{X} f(\omega)(x)u(x)d\mu(\omega)$.

Savage's Framework

Some Definitions

- $E \subseteq \Omega$: an event;
- fEg: **conditional act**, s.t. for acts f, g and event E, $fEg \in \mathcal{F} : (fEg)(\omega) = f(\omega)$ if $\omega \in E$ and $(fEg)(\omega) = g(\omega)$ if ow;
- **Null event** E: event s.t. $\forall f, g, h \in \mathcal{F} : \succ g, fEh \sim gEh$;
- $\tilde{x}$: **constant act**, $\tilde{x}(\omega) = x, \forall \omega \in \Omega$.

Savage's Framework

Postulates

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(P2 gives a form of independence.)

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(P3 allows us to rank acts based on the ranking of constant acts.)

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(P3 allows us to rank acts based on the ranking of constant acts.)

P4 (Weak Comparative Probability): $\forall$ events A, B and constant acts $\tilde{x}, \tilde{x}', \tilde{y}, \tilde{y}'$, s.t.
$\tilde{x} \succ \tilde{y}$ and $\tilde{x}' \succ \tilde{y}'$,
$\tilde{x}A\tilde{y} \succsim \tilde{x}B\tilde{y} \iff \tilde{x}'A\tilde{y}' \succsim \tilde{x}'B\tilde{y}'$.
(P4 is crucial to infer from preferences alone whether an event A is more likely than another event B. Clearly separates taste and belief.)

Savage's Framework

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(P6 is a form of Archimedean property. This indirectly imposes constraints on $\boldsymbol{\Omega}$.)

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(P6 is a form of Archimedean property. This indirectly imposes constraints on Ω.)

P7 (Uniform Monotonicity): $\forall$ event E and acts f, g, (i) if $fEh \succ \tilde{g}(\omega)Eh$ for any $\omega \in E$ – i.e., $\tilde{g}(\omega)$ is constant act equal to $g(\omega) = x$ in every state ω' – and any act h, then $fEh \succsim gEh$; (ii) if $\tilde{f}(\omega)Eh \succ gEh \; \forall \omega \in E$, then $fEh \succsim gEh$.

Savage's SEU

Theorem

$\succsim$ satisfies P1-P7 if and only if there exist

(i) unique, nonatomic, finitely additive $\mu \in \Delta(\Omega)$ s.t. $\mu(E) = 0 \iff E$ is null event;

(ii) $u : X \to \mathbb{R}$, bounded and unique up to positive affine transformations

s.t. $\forall f, g \in \mathcal{F}$,

$$f \succsim g \quad \iff \quad \mathbb{E}_\mu[u \circ f] := \int_\Omega u(f(\omega))d\mu(\omega) \geq \int_\Omega u(g(\omega))d\mu(\omega) = \mathbb{E}_\mu[u \circ g].$$

Overview

1. Uncertainty
2. Subjective Expected Utility
3. Savage's Framework
4. Uncertainty Aversion
   - Ellsberg Paradox
   - A Set of Probability Measures: Maxmin Expected Utility
   - Beliefs over Unknown Probabilities
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Most people choose *A*.

Ellsberg Paradox

A box contains 60 balls: 20 are black and the rest are either red or green. Which would you prefer:

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This is incompatible with SEU. (Why?)

Ellsberg Paradox

Independence always gets the blame. Ways of dealing with it:

- Maxmin EU and 'sets of priors' (Gilboa & Schmeidler 1989 JMathEcon)
- Choquet EU and capacities instead of prob. measures (Schmeidler 1989 Ecta)
- Uncertainty Aversion (Klibanoff, Marinacci, & Mukerji 2005 Ecta; Denti & Pomatto 2022 Ecta)

Maxmin Expected Utility

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It also implies DM is indifferent between f, g, and $\tilde{q}$, where $q := 1/2p + 1/2\delta_0$.

Indeed, it is not unreasonable to consider that $\tilde{q} \succ f \sim g$, as $\tilde{q}$ entails no uncertainty.

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$\succsim \subseteq \mathcal{F}^2$ is **GS uncertainty averse** (neutral/seeking) if $\forall f, g \in \mathcal{F}, f \sim g \implies \frac{1}{2}f + \frac{1}{2}g \succsim f$ $(\sim / \precsim)$.

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$\succsim \subseteq \mathcal{F}^2$ sat. **C-independence** if $\forall f, g \in \mathcal{F}$, $p \in \Delta(X)$, and $\alpha \in (0, 1]$, $f \succsim g \iff \alpha f + (1 - \alpha)\tilde{p} \succsim \alpha g + (1 - \alpha)\tilde{p}$.

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Hedging is only valuable when it can eliminate uncertainty, which is not the case if it uses a constant act.

Maxmin Expected Utility

Theorem

Let $\succsim$ be preference relation on $\mathcal{F}$. $\succsim$ satisfies continuity, monotonicity, C-independence, and GS uncertainty aversion if and only if

$\exists u : X \to \mathbb{R}$ and convex and compact set $\mathcal{M} \subseteq \Delta(\Omega)$ s.t.

$$f \succsim g \quad \iff \quad \min_{\mu \in \mathcal{M}} \mathbb{E}_\mu[\mathbb{E}_f[u]] \geq \min_{\mu \in \mathcal{M}} \mathbb{E}_\mu[\mathbb{E}_g[u]].$$

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Different $\succsim$ can induce representations with different $\mathcal{M}$.

Maxmin implicitly assumes extreme uncertainty aversion, behaving as if expecting worst to happen among all prob. distr. they entertain.

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$$U(f) := \int_{\Delta(\Omega)} \phi \left(\int_{\Omega} u(f(\omega)) d\mu(\omega) \right) d\pi(\mu)$$

Interpretation

- $f : \Omega \to X$ is Savage act
- $u : X \to \mathbb{R}$ vNM utility
- $\phi : \mathbb{R} \to \mathbb{R}$ a strictly increasing and continuous function
- $\mu \in \Delta(\Omega)$ is a prob. measure on state space
- $\pi \in \Delta(\Delta(\Omega))$ is DM's *prior*, capturing uncertainty about how state is actually distributed.

Smooth Uncertainty Aversion:

- curvature of u captures risk attitudes
- curvature of ϕ captures uncertainty attitudes (concave/linear/convex)

  Maxmin as limit of extreme risk aversion.

Overview

More!

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- **Relation between risk and uncertainty attitudes** (Halevy 2007 Ecta; also Chapman et al. 2023 JPE Micro): attitudes to ambiguity and compound objective lotteries are tightly associated
- **Methods to elicit beliefs and patterns in belief updating:** important beyond just experimental and theory! E.g., development and education (Dizon-Ross 2019 AER), macro (Bordalo et al. 2020 AER), health (de Paula, Valente, & Miller 2022 WP), finance (Giglio et al. 2021 AER), political economy (Ortoleva Snowberg 2015 AER) Take + theory and experimental!

Where does this leave SEU?

SEU remains a benchmark framework: very appealing principles and well-known virtues and vices.

Behaviourally: neither comes for free and it's important to know this.

Model is approximation and, unless there is a crucial element missing, SEU are defaults so as to better understand differences in the model (i.e., what's the effect of the new ingredient on the soup's flavour overall).